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GENERALIZED SCHUR OPERATORS ON PLANAR BINARY TREES

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ABSTRACT. We introduce new families of operators on the vector space spanned by rooted planar binary trees. We prove that they are generalized Schur operators. For this purpose, we construct a correspondence, which is an extension of Fomin's r-correspondence.

1. INTRODUCTION

Young's lattice is a prototypical example of differential posets introduced by Stanley [11, 12]. A standard Young tableau can be identified with a path in Young's lattice. Under this identification, the Robinson correspondence is a bijection between permutations and some pairs of paths in Young's lattice. The correspondence was generalized for differential posets, and further for dual graphs (generalizations of differential posets [2]) by Fomin [1, 3] (see also [10]). His method is as follows. The up and down operators U and D of r-dual graphs satisfy the relation $DU - UD = rI$, which implies some local correspondences, called r-correspondences. By piecing them together, we can construct global correspondences, which are the Robinson correspondence in special cases. In this sense, paths in differential posets or dual graphs are analogues of standard Young tableaux with respect to the Robinson correspondence.

A bijection between certain matrices and pairs of semi-standard tableaux is known as the Robinson-Schensted-Knuth correspondence. In [4], Fomin introduced generalized Schur operators, and generalized the method of the Robinson correspondence to that of the Robinson-Schensted-Knuth correspondence. Roughly speaking, generalized Schur operators are collections of up and down operators with some commutation relations (see Definition 2.1). The relations mean some local correspondences, which are extensions of r-correspondences. Again, by piecing them together, we can construct global correspondences, which are the Robinson-Schensted-Knuth correspondence in special cases. Equivalently, the Robinson-Schensted-Knuth correspondence, one of the most important combinatorial properties of semi-standard Young tableaux, is induced from the relations of generalized Schur operators.

In this paper, we consider the vector space spanned by rooted planar binary trees. We introduce new families of linear operators on the space, which have a relationship with some labellings on rooted planar binary trees. We show that they are generalized Schur operators by constructing an extension of an r-correspondence. As applications, we can generalize the Loday-Ronco correspondence, which is a bijection between permutations and pairs of labeling on planar binary trees, by the Fomin's method. In addition, we can show Pieri formula and Cauchy identity for weighted generating functions of labellings on planar binary trees. Those generating functions are commutativizations of basis elements of the Hopf algebra called Loday-Ronco algebra.

2. PRELIMINARIES

In this section, we define our main objects. In Subsection 2.1, we recall the definition of generalized Schur operators introduced by Fomin [4]. We recall the definition of rooted planar binary trees and labellings on them in Subsection 2.2, and then we introduce linear operators on the vector space whose basis is the set of rooted planar binary trees in Subsection 2.3.

2.1. Generalized Schur Operators.

Let K be a field of characteristic zero that contains all formal power series of variables $t, t', t_1, t_2, \ldots$ Let V_i be finite-dimensional K-vector spaces for all $i \in \mathbb{Z}$. Fix a basis Y_i of each V_i so that $V_i = KY_i$. Let $Y = \coprod_i Y_i$, $V = \bigoplus_i V_i$ and $\widehat{V} = \prod_i V_i$. For a sequence $\{A_i\}$ and a formal variable x, $A(x)$ denotes the generating function $\sum_{i\geq 0} A_i x^i$.

Definition 2.1. We call $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ *generalized Schur operators with* $\{a_m\}$ if the following conditions are satisfied:

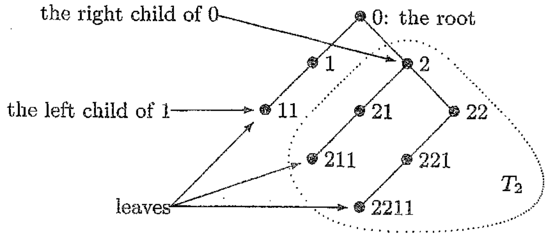
- $\{a_m\}$ is a sequence of elements of K.
- U_i is a linear map on V satisfying $U_i(V_j) \subset V_{j+i}$ for all j.
- D_i is a linear map on V satisfying $D_i(V_j) \subset V_{j-i}$ for all j.
- The equation $D(t')U(t) = a(tt')U(t)D(t')$ holds.

2.2. Rooted Planar Binary Trees.

Let F be the monoid of words generated by the alphabet $\{1, 2\}$, and let 0 denote the word whose length is zero. We also regard F as a poset by $v \leq vw$ for $v, w \in F$. We call a subset $T \subset F$ an *ideal* of the poset F if $w \leq v$ for some $v \in T$ implies $w \in T$. We call a finite ideal of the poset F a *rooted planar binary tree* or shortly *tree*. Let $\mathbb{T}$ denote the set of trees.

Let T be a tree. An element of T is called a *node* of T. Let $\mathbb{T}_i$ be the set of trees of i nodes. For a node v, we call the node $v2$ (resp. $v1$) the *right* (resp. *left*) *child* of v. A node without children is called a *leaf*. If T is nonempty, $0 \in T$. We call 0 the *root* of T. For $T \in \mathbb{T}$ and $v \in F$, we define T_v by $T_v := \{\, w \in T \mid v \leq w \,\}$.

Example 2.2. Let $T = \{ 0, 11, 2, 21, 211, 22, 221, 2211 \}$. Then we have



Definition 2.3. Let T be a tree, and m a positive integer. We call a map $\varphi\colon T \to \{ 1, \dots, m \}$ a right-strictly-increasing labelling if

- $\varphi(w) \leq \varphi(v)$ for $w \in T$ and $v \in T_{w1}$, and
- $\varphi(w) < \varphi(v)$ for $w \in T$ and $v \in T_{w2}$.

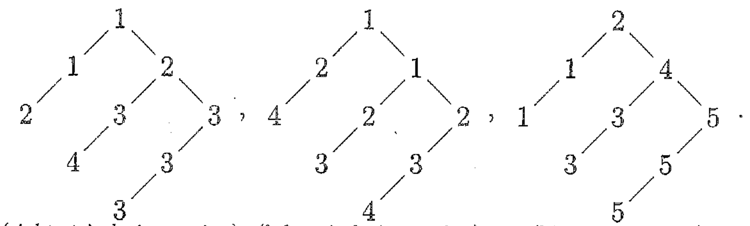
We call a map $\phi\colon T \to \{ 1, \dots, m \}$ a left-strictly-increasing labelling if

- $\phi(w) < \phi(v)$ for $w \in T$ and $v \in T_{w1}$, and
- $\phi(w) \leq \phi(v)$ for $w \in T$ and $v \in T_{w2}$.

We call a map $\psi\colon T \to \{ 1, \dots, m \}$ a binary-searching labelling if

- $\psi(w) \geq \psi(v)$ for $w \in T$ and $v \in T_{w1}$, and
- $\psi(w) < \psi(v)$ for $w \in T$ and $v \in T_{w2}$.

Example 2.4. The following labellings are respectively right-strictly-increasing, left-strictly-increasing, and binary-searching:



(right-strictly-increasing) (left-strictly-increasing) (binary-searching)

2.3. **Definition of our generalized Schur operators.** In this subsection, we define linear operators U_i, U'_i, and D_i. In Section 3, we shall show that they are generalized Schur operators.

2.3.1. *Up operators.* We define up operators U_i (resp. U'_i) and consider a relation between the up operators U_i (resp. U'_i) and right-strictly (resp. left-strictly) labellings.

Definition 2.5. We define the edges G_U of oriented graphs whose vertices are trees to be the set of pairs (T, T') of trees satisfying the following:

- $T \subset T'$.
- For each $w \in T' \setminus T$, there exists $v_w \in T$ such that $w = v_w 1^n$ or $w = v_w 21^n$ for some nonnegative integer n if $T \neq \emptyset$.

- For each $w \in T' \setminus T$, $w = 1^n$ for some nonnegative integer n if $T = \emptyset$.

We call T' *a tree obtained from T by adding some l-strips* if $(T, T') \in G_U$. For $i \in \mathbb{N} = \{ 0, 1, 2, \dots \}$, we define G_{U_i} by

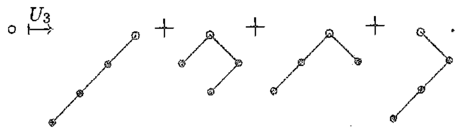
$$G_{U_i} = \{ (T, T') \in G_U \mid |T| + i = |T'| \} .$$

Definition 2.6. For $i \in \mathbb{N}$ and $T \in \mathbb{T}$, we define linear operators U_i on $K\mathbb{T}$ by

$$U_i T = \sum_{T' : (T, T') \in G_{U_i}} T'.$$

Equivalently, $U_i T$ is the sum of all trees obtained from T by adding l-strips with i nodes.

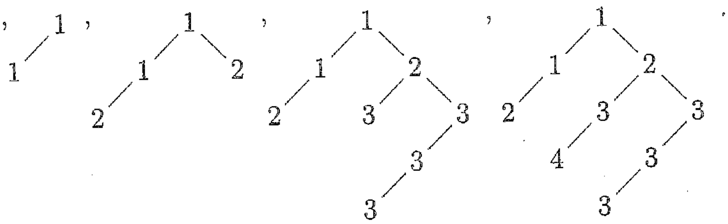
Example 2.7. The action of U_3 on $\{ 0 \}$ is as follows:



Remark 2.8. Let φ be a right-strictly-increasing labelling. The inverse image $\varphi^{-1}(\{1, \dots, n+1\})$ is the tree obtained from the inverse image $\varphi^{-1}(\{1, \dots, n\})$ by adding l-strips. Hence we identify right-strictly-increasing labellings with paths $(\emptyset = T^0, T^1, \dots, T^m)$ of the graph $(\mathbb{T}, G_U)$.

Example 2.9. We identify the right-strictly-increasing labelling in Example 2.4 with the sequence

$\emptyset$,



Next we define another family of up operators U'_i.

Definition 2.10. We define the edges $G_{U'}$ of oriented graphs whose vertices are trees to be the set of pairs (T, T') of trees satisfying the following:

- $T \subset T'$.
- For each $w \in T' \setminus T$, there exists $v_w \in T$ such that $w = v_w 2^n$ or $w = v_w 12^n$ for some nonnegative integer n if $T \neq \emptyset$.
- For each $w \in T' \setminus T$, $w = 2^n$ for some nonnegative integer n if $T = \emptyset$.

We call T' *a tree obtained from T by adding r-strips* if $(T,T') \in G_{U'}$. For $i \in \mathbb{N}$, we define $G_{U'_i}$ by

$$G_{U'_i} = \{ (T,T') \in G_U \mid |T| + i = |T'| \}.$$

Definition 2.11. For $i \in \mathbb{N}$ and $T \in \mathbb{T}$, we define linear operators U'_i on $K\mathbb{T}$ to be

$$U'_i T = \sum_{T' : (T,T') \in G_{U'_i}} T'.$$

Equivalently, $U'_i T$ is the sum of all trees obtained from T by adding r-strips with i nodes.

Remark 2.12. We identify left-strictly-increasing labellings with paths $(\emptyset = T^0, T^1, \ldots, T^m)$ of the graph $(\mathbb{T}, G_{U'})$.

2.3.2. *Down operators.* We define down operators D_i on $K\mathbb{T}$, and we consider relations between the down operators D_i and binary searching labellings.

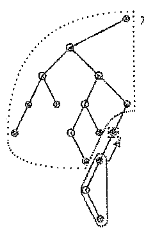
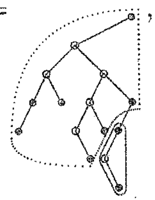
For $T \in \mathbb{T}$, let R_T denote the set $\{w \in T | w2 \notin T\}$. For $w \in R_T$, we define

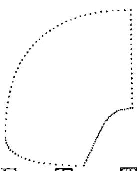
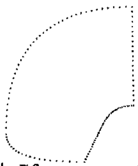
$$T \ominus w = (T \setminus T_w) \cup \{ wv \mid w1v \in T_w \}.$$

There exists the inclusion $\nu_{T,w}$ from $T \ominus w$ to T defined by

$$\begin{cases} \nu_{T,w}(wv) = w1v & (wv \in T_w) \\ \nu_{T,w}(v') = v' & (v' \notin T_w). \end{cases}$$

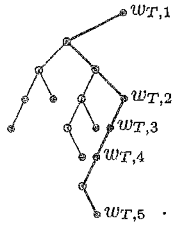
Example 2.13. For $w = 1221$ and

$T =$ , $\quad T \ominus w =$ ,

where ⊛ are nodes in R_T or $R_{T \ominus w}$, and ⊛ is $w = 1221$. The inclusion $\nu_{T,w}$ maps the nodes in  of $T \ominus w$ to the nodes in  of T, and the nodes in  of $T \ominus w$ to the nodes in  of T, respectively.

For $T \in \mathbb{T}$, let E_T denote $\{w \in T | \text{ If } w = v1w' \text{ then } v2 \notin T\}$. Roughly speaking, it is the set of nodes of T between the root 0 and the right-most node of T. We define r_T by $r_T = E_T \cap R_T$. The set r_T is a chain. Let $r_T = \{w_{T,1} < w_{T,2} < w_{T,3} < \cdots < w_{T,k}\}$. Let $r_{T,i}$ denote the ideal $\{w_{T,1}, w_{T,2}, w_{T,3}, \ldots, w_{T,i}\}$ of r_T consisting of i nodes.

Example 2.14. Let T be the one in Example 2.13. Then the nodes in E_T are on the thick line, and the nodes in R_T are $\bullet$ in the following picture:



Hence $r_T = \{ 0, 122, 1221, 12211, 1221112 \}$, and $r_{T,3} = \{ 0, 122, 1221 \}$.

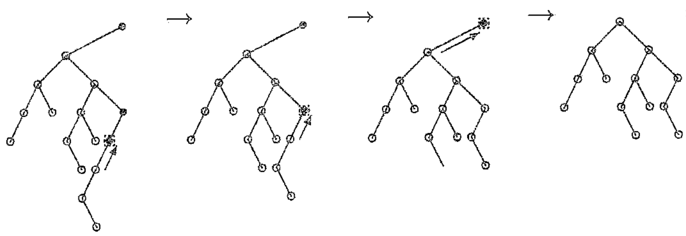
We define $T \ominus r_{T,i}$ inductively by

$$\begin{cases} (T \ominus w_{T,i}) \ominus r_{T,i-1} & i > 0 \\ T & i = 0. \end{cases}$$

We also define the inclusion $\nu_{T,i}$ from $T \ominus r_{T,i}$ to T inductively by

$$\nu_{T,i} = \nu_{T \ominus w_{T,i}, i-1} \circ \nu_{T, w_{T,i}}.$$

Example 2.15. For T in Example 2.13, we have



$= T \ominus r_{T,3}.$

The inclusion $\nu_{T,3}$ maps the nodes $\circ$ in $T \ominus r_{T,3}$ to the nodes $\circ$ in T.

We also define a bijection $\widetilde{\nu}_{T,i}$ from the words F of $\{ 1, 2 \}$ to $F \setminus r_{T,i}$ by

$$\widetilde{\nu}_{T,i}(w) = \nu_{T,i}(v)v',$$

where $w = vv'$ and $v = \max\{ u \in T \ominus r_{T,i} \mid w = uu' \}$. By definition, $\widetilde{\nu}_{T,i}(w) = \nu_{T,i}(w)$ for $w \in T \ominus r_{T,i}$.

Definition 2.16. We define the edges G_D of oriented graphs whose vertices are trees to be the set of pairs (T, T') of trees such that $T = T' \ominus r_{T',i}$ for some i. For $i \in \mathbb{N}$, we define G_{D_i} by

$$G_{D_i} = \{ (T, T') \in G_D \mid |T| + i = |T'| \}.$$

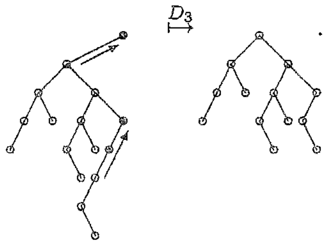
Remark 2.17. By definition, $G_{D_0} = \{ (T, T) \mid T \in \mathbb{T} \}$. For each i and $T \in \mathbb{T}$, $|\{ (T', T'') \in G_{D_i} \mid T'' = T \}| \leq 1$.

Definition 2.18. For $i \in \mathbb{N}$, we define linear operators D_i by

$$D_i T = \sum_{T' :\, (T',T) \in G_{D_i}} T'$$

for $T \in \mathbb{T}$.

Example 2.19. The operator D_3 acts as follows:



Next we consider a relation between G_D and binary-searching labellings. Let $\psi_m : T \to \{1, \ldots, m\}$ be a binary-searching labelling. By the definition of binary-searching labelling, the inverse image $\psi_m^{-1}(\{m\})$ equals $r_{T,k_m} = \{w_{T,1}, \ldots, w_{T,k_m}\}$ for some k_m. Hence we can construct the tree $T \ominus \psi_m^{-1}(\{m\})$. Let T^{m-1} be the tree $T \ominus \psi_m^{-1}(\{m\})$. Then the inclusion ν_{T,k_m} induces a binary-searching labelling

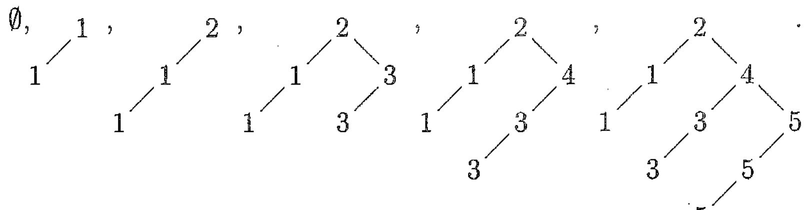
$$\psi_{m-1} = \psi_m \circ \nu_{T,k_m} : T^{m-1} \to \{1, \ldots, m-1\}$$

on the tree T^{m-1}. Hence we identify binary-searching labellings on T with paths

$$(\emptyset = T^0, T^1, \ldots, T^m = T)$$

of the graph $(\mathbb{T}, G_D)$.

Example 2.20. We identify the binary-searching labelling in Example 2.4 with the sequence



3. MAIN RESULTS

We retain all the notation used in the previous sections. In Subsection 3.1, we show the main theorems, which are proved in Subsection 3.2.

3.1. **Main Results.**

Theorem 3.1. *The operators $D(t)$ and $U(t')$ satisfy the equation*

$$D(t)U(t') = \frac{1}{1-tt'}U(t')D(t). \tag{1}$$

Equivalently, $D(t_1)\cdots D(t_n)$ and $U(t_n)\cdots U(t_1)$ are generalized Schur operators with $\{1,1,1,\ldots\}$.

Theorem 3.2. *The operators $D(t)$ and $U'(t')$ satisfy the equation*

$$D(t)U'(t') = (1+tt')U'(t')D(t). \tag{2}$$

Equivalently, $D(t_1)\cdots D(t_n)$ and $U'(t_n)\cdots U'(t_1)$ are generalized Schur operators with $\{1,1,0,0,0,0\ldots\}$.

Corollary 3.3. *The graphs $(\mathbb{T}, G_{U_1} = G_{U'_1})$ and $(\mathbb{T}, G_{D_1})$ are 1-dual in the sense of Fomin [4]. Equivalently, U_1 and D_1 satisfy the equation*

$$D_1U_1 - U_1D_1 = I,$$

where I is the identity map on V.

Remark 3.4. Nzeutchap [8] constructs r-dual graphs from dual Hopf algebras. The graphs $(\mathbb{T}, G_{U_1})$ and $(\mathbb{T}, G_{D_1})$ are identified with the 1-dual graphs obtained from the Loday-Ronco algebra by his method.

Corollary 3.5. *The up and down operators U_1 and D satisfy*

$$DU_1 - U_1D = D.$$

Let $\langle\ ,\ \rangle$ be the natural pairing in KY, i.e., the bilinear form on $\widehat{V} \times V$ such that $\langle \sum_{\lambda\in Y} a_\lambda \lambda, \sum_{\mu\in Y} b_\mu \mu\rangle = \sum_{\lambda\in Y} a_\lambda b_\lambda$. We define U_i^* and D_i^* as the maps obtained from the adjoints of U_i and D_i with respect to $\langle\ ,\ \rangle$ by restricting to V, respectively. Then we have the following corollary.

Corollary 3.6. *The up and down operators D_1^* and U^* satisfy*

$$U^*D_1^* - D_1^*U^* = U^*,$$

where $U^* = \sum_i U_i^*$.

3.2. **Proof of Main results.** In this subsection, we prove Theorems 3.1 and 3.2. First we rewrite their statements as the equations of the cardinalities of some sets (Remark 3.10). Then we show the equations by constructing bijections (Lemmas 3.11 and 3.12).

Lemma 3.7. *The equation (1) is equivalent to*

$$D_jU_i = \sum_{k=0}^{\min(i,k)} U_{i-k}D_{j-k} \qquad \text{for all } i,j. \tag{3}$$

Lemma 3.8. *The equation (2) is equivalent to*

$$D_jU'_i = \sum_{k=0}^{\min(1,i,k)} U'_{i-k}D_{j-k} \qquad \textit{for all } i,j. \tag{4}$$

Definition 3.9. We define $N_{i,j}(T,T')$ and $N'_{i,j}(T,T')$ by

$$N_{i,j}(T,T') = \{\ ((T,T''),(T',T'')) \in G_{U_i} \times G_{D_j}\ \}$$

and

$$N'_{i,j}(T,T') = \{\ ((T,T''),(T',T'')) \in G_{U'_i} \times G_{D_j}\ \}.$$

We define $S_{j,i}(T,T')$ and $S'_{j,i}(T,T')$ by

$$S_{j,i}(T,T') = \{\ ((T'',T),(T'',T')) \in G_{D_j} \times G_{U_i}\ \}$$

and

$$S'_{j,i}(T,T') = \{\ ((T'',T),(T'',T')) \in G_{D_j} \times G_{U'_i}\ \}.$$

We also define $\widetilde{S}_{j,i}(T,T')$ and $\widetilde{S}'_{j,i}(T,T')$ by

$$\widetilde{S}_{j,i}(T,T') = \coprod_{k=0}^{\min(i,j)} S_{j-k,i-k}(T,T')$$

$$\widetilde{S}'_{j,i}(T,T') = \coprod_{k=0}^{\min(1,i,j)} S'_{j-k,i-k}(T,T'),$$

where $\coprod$ denotes the disjoint union.

Roughly speaking, $N_{i,j}(T,T')$ and $N'_{i,j}(T,T')$ are the set of pairs of edges which share the same trees as their end points, while $S_{j,i}(T,T')$ and $S'_{j,i}(T,T')$ are the set of pairs of edges which share the same trees as their start points.

Remark 3.10. By definition,

$$\langle D_jU_iT,T'\rangle = |N_{i,j}(T,T')|,$$
$$\langle D'_jU_iT,T'\rangle = |N'_{i,j}(T,T')|,$$
$$\langle U_jD_iT,T'\rangle = |S'_{i,j}(T,T')|,$$

and

$$\langle U_jD'_iT,T'\rangle = |S'_{i,j}(T,T')|$$

for $T,T' \in \mathbb{T}$. Hence the equation (3) (resp. (4)) is equivalent to the equation $|N_{i,j}(T,T')| = |\widetilde{S}_{j,i}(T,T')|$ (resp. $|N'_{i,j}(T,T')| = |\widetilde{S}'_{j,i}(T,T')|$).

Lemma 3.11. *For T, $T' \in \mathbb{T}$ and i, $j \in \mathbb{N}$, there exists a bijection from $N_{i,j}(T,T')$ to $\widetilde{S}_{j,i}(T,T')$.*

Proof. First we construct an element of $\widetilde{S}_{j,i}(T,T')$ from an element of $N_{i,j}(T,T')$. Let $((T,T''),(T',T''))$ be an element of $N_{i,j}(T,T')$. Equivalently, (T,T'') is an edge of G_{U_i} such that $T'' \ominus r_{T'',j} = T'$. Let k be $j - |r_{T'',j} \cap r_T|$. We have $r_{T,j-k} = r_{T'',j} \cap r_T$ since $r_{T''}$ is one of the following:

$$r_T,$$
$$r_{T,l} \cup \{\, w_{T,l+1}21^i \,|\, i \le n \,\},$$
$$r_{T,l} \cup \{\, w_{T,l+1}1^i \,|\, i \le n \,\}$$

for some l, $n \in \mathbb{N}$. Let us consider

$$((T \ominus r_{T,j-k}, T), (T \ominus r_{T,j-k}, T')).$$

We prove $((T \ominus r_{T,j-k}, T), (T \ominus r_{T,j-k}, T')) \in \widetilde{S}_{j,i}(T,T')$. Equivalently, we prove $(T \ominus r_{T,j-k}, T) \in G_{D_{j-k}}$ and $(T \ominus r_{T,j-k}, T') \in G_{U_{i-k}}$. By definition, it is clear that the edge $(T \ominus r_{T,j-k}, T)$ is in $G_{D_{j-k}}$. On the other hand, $(T,T'') \in G_{U_i}$ implies that $(T \ominus r_{T,j-k}, T'' \ominus r_{T'',j})$ is in $G_{U_{i-k}}$. Since $T' = T'' \ominus r_{T'',j}$, the edge $(T \ominus r_{T,j-k}, T')$ is in $G_{U_{i-k}}$. Hence we have $((T \ominus r_{T,j-k}, T), (T \ominus r_{T,j-k}, T')) \in \widetilde{S}_{j,i}(T,T')$.

Next we construct an element of $N_{i,j}(T,T')$ from an element of $\widetilde{S}_{j,i}(T,T')$. Let $((T''',T),(T''',T'))$ be an element of $\widetilde{S}_{j,i}(T,T')$. Equivalently, (T''',T') is an edge of G_{U_i-k} such that $T \ominus r_{T,j-k} = T'''$. First we consider the case where $|r_T| > j-k$. Let $\omega = w_{T,j-k+1}$ and $\omega' \in \nu^{-1}_{T,j-k}(\omega)$. Since $\omega' \in T \ominus r_{T,j-k}$ and $\omega'2 \notin T \ominus r_{T,j-k}$, we have

$$T'_{\omega'2} = \{\, \omega'2, \omega'21, \dots, \omega'21^{n-1} \,\}$$

for some $n \in \mathbb{N}$. (In the case where $n = 0$, $\{\, \omega'2, \omega'21, \dots, \omega'21^{n-1} \,\}$ denotes the empty set.) For such n, let R denote

$$\{\, \omega2, \omega21, \dots, \omega21^{n-1+k} \,\}.$$

We define T'' to be

$$\widetilde{\nu}_{T,j-k}(T) \cup r_{T,j-k} \cup R.$$

Since $r_{T''} = r_{T,j-k} \cup R$, the pair $((T,T''),(T',T''))$ is in $N_{i,j}(T,T')$. Next we consider the case where $|r_T| = j-k$. Let ω be

$$\max\{\, w \notin r_T \,|\, w < w_{T,j-k} \,\}$$

and $\omega' \in \nu^{-1}_{T,j-k}(r)$. Since $\omega' \in T \ominus r_{T,j-k}$ and $\omega'2 \notin T \ominus r_{T,j-k}$, we have

$$T'_{\omega'2} = \{\, \omega'2, \omega'21, \dots, \omega'21^{n-1} \,\}$$

for some $n \in \mathbb{N}$. For such n, let R denote

$$\{\, w_{T,j-k+1}1, \dots, w_{T,j-k+1}1^{n-1+k} \,\}.$$

We define T'' to be

$$\widetilde{\nu}_{T,j-k}(T) \cup r_{T,j-k} \cup R.$$

Since $r_{T''} = r_{T,j-k} \cup R$, the pair $((T,T''),(T',T''))$ is in $N_{i,j}(T,T')$. Thus we can construct an element of $N_{i,j}(T,T')$ from an element of $\widetilde{S}_{j,i}(T,T')$.

By the definition of them, these constructions are the inverses of each other. Hence we have the lemma. □

We can prove Lemma 3.12 by the same argument as in the proof of Lemma 3.11.

Lemma 3.12. *For T, $T' \in \mathbb{T}$, there exists a bijection from $N'_{i,j}(T,T')$ to $\widetilde{S}'_{j,i}(T,T')$.*

Lemmas 3.11 and 3.12 imply Theorems 3.1 and 3.2.

4. Application

In this section, we consider a relation between our generalized Schur operators and the Loday-Ronco algebra.

We have correspondences between the sets $N_{i,j}(T,T')$ and $\widetilde{S}_{i,j}(T,T')$ for all i,j by the proof of Lemma 3.11. From them, we can construct a Robinson-Schensted-Knuth correspondence for paths of G_U and G_D by the method in [4]. This correspondence is a generalization of the Loday-Ronco correspondence, which is a Robinson correspondence for labellings on binary trees. By Lemma 3.12, we also have correspondences between $N'_{i,j}(T,T')$ and $\widetilde{S}'_{j,i}(T,T')$. By the same argument as in the case of G_U and G_D, we can construct a Robinson-Schensted-Knuth correspondence for paths of $G_{U'}$ and G_D, which is another generalization of the Loday-Ronco correspondence.

Remark 4.1. Rey gave a construction of the Loday-Ronco algebra in [9]. He introduced a new Robinson-Schensted-Knuth correspondence for binary trees to construct the Loday-Ronco algebra. Some of our correspondences are equivalent to his correspondence.

Definition 4.2. For $\lambda,\mu \in V$, we define quasi-symmetric polynomials $s^D_{\lambda,\mu}(t_1,\dots,t_n)$, $s^{\lambda,\mu}_U(t_1,\dots,t_n)$, and $s^{\lambda,\mu}_{U'}(t_1,\dots,t_n)$ by

$$s^D_{\lambda,\mu}(t_1,\dots,t_n) = \langle D(t_1)\cdots D(t_n)T,T'\rangle\,,$$
$$s^{\lambda,\mu}_U(t_1,\dots,t_n) = \langle U(t_n)\cdots U(t_1)T',T\rangle\,,$$
$$s^{\lambda,\mu}_{U'}(t_1,\dots,t_n) = \langle U'(t_n)\cdots U'(t_1)T',T\rangle\,.$$

For a labelling φ from T to $\{1,\dots,m\}$, set $t^\varphi = \prod_{w\in T} t_{\varphi(w)}$. For a tree T, by the definition of labellings,

$$s^D_{T,\emptyset}(t_1,\dots,t_n) = \sum_\psi t^\psi,$$
$$s^{T,\emptyset}_U(t_1,\dots,t_n) = \sum_\varphi t^\varphi,$$

$$s_{U'}^{T,\emptyset}(t_1,\dots,t_n)=\sum_{\phi} t^{\phi},$$

where the first sum is over all binary-searching labellings ψ on T, the second over all right-strictly-increasing labellings φ on T, and the last over all left-strictly-increasing labellings ϕ on T.

Remark 4.3. The polynomials $s_U^{T,\emptyset}(t_1,\dots,t_n)$ and $s_{T,\emptyset}^{D}(t_1,\dots,t_n)$ are the commutativizations of the basis elements $\mathbb{Q}_T$ and $\mathbb{P}_T$ of $\mathbf{PBT}$ in Hivert-Novelli-Thibon [6].

Since $D(t)$ and $U(t)$ are generalized Schur operators, we have Pieri formula for $s_U^{T,\emptyset}(t_1,\dots,t_n)$ and $s_{T,\emptyset}^{D}(t_1,\dots,t_n)$ by [7]. By [4], we have Cauchy identity for them. We also have a "skew" version of them. We also have Pieri formula and Cauchy identity for $s_{U'}^{T,\emptyset}(t_1,\dots,t_n)$ and $s_{T,\emptyset}^{D}(t_1,\dots,t_n)$.

Remark 4.4. These polynomials are not symmetric in general. This is because D_i does not commute with D_j in general. For example, since

$$\begin{aligned}
&D(t_1)D(t_2)\{\,0,1,12\,\}\\
&=D(t_1)(\{\,0,1,12\,\}+t_2\{\,0,2\,\}+t_2^2\{\,0\,\})\\
&=(\{0,1,12\}+t_1\{0,2\}+t_1^2\{0\})+t_2(\{0,2\}+t_1\{0\})+t_2^2(\{0\}+t_1\emptyset),
\end{aligned}$$

we have $\langle D(t_1)D(t_2)\{\,0,1,12\,\},\emptyset\rangle=t_1t_2^2$, which is not symmetric.

REFERENCES

[1] S. Fomin, *Generalized Robinson-Schensted-Knuth correspondence*, Zariski Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986), 156–175, 195 (Russian); English transl., J. Soviet Math. 41(1988), 979–991.

[2] S. Fomin, *Duality of graded graphs*, J. Algebraic Combin. **3** (1994), 357–404.

[3] S. Fomin, *Schensted algorithms for dual graded graphs*, J. Algebraic Combin. **4** (1995), 5–45.

[4] S. Fomin, *Schur operators and Knuth correspondences*, J. Combin. Theory, Ser. A **72** (1995), 277–292.

[5] Ira M. Gessel, *Counting paths in Young's lattice*, J. Statistical planning and inference. **34** (1993), 125–134.

[6] F. Hivert, J. Novelli and J. Thibon, *The algebra of binary search trees*, Theor. Comput. Sci. 339, 1 (Jun. 2005), 129–165.

[7] Y. Numata, *Pieri's Formula for Generalized Schur Polynomials*, J. Algebr. Comb. **26** (2007), 27–45.

[8] J. Nzeutchap, Graded Graphs and Fomin's r-correspondences associated to the Hopf Algebras of Planar Binary Trees, Quasi-symmetric Functions and Noncommutative Symmetric Functions, FPSAC '06, 2006. `http://garsia.math.yorku.ca/fpsac06/papers/53.pdf`

[9] M. Rey, A new construction of the Loday-Ronco algebra, FPSAC '06, 2006. `http://garsia.math.yorku.ca/fpsac06/papers/51.pdf`

[10] T. Roby, *Applications and extensions of Fomin's generalization of the Robinson-Schensted correspondence to differential posets*, Ph.D. thesis, M.I.T., 1991.

[11] R. Stanley, *Differential posets*, J. American Math. Soc, 1 (1988), 919–961.
[12] R. Stanley, *Variations on differential posets*, Invariant theory and tableaux (Stanton,D.,ed.), IMA volumes in mathematics and its applications, Springer-Verlag, New York, 145–165.

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